**FACULTY OF MATHEMATICS AND PHYSICS**
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**HABILITATION THESIS**

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# Lipschitz-free Banach spaces

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# Contents

# 1 Preface

This thesis presents the following eight recent papers coauthored by Marek Cúth:

- [**A**] M. Cúth, M. Doucha: Lipschitz-Free Spaces Over Ultrametric Spaces, Mediterr. J. Math., 13 (2016), 1893-1906, doi: 10.1007/s00009-015-0566-7.
- [**B**] M. Cúth, M. Doucha, P. Wojtaszczyk: On the structure of Lipschitz-free spaces, Proc. Amer. Math. Soc., 144 (9) (2016), 3833–3846, doi: 10.1090/proc/13019.
- [**C**] M. Cúth, M. Johanis: Isometric embedding of $\ell_1$ into Lipschitz-free spaces and $\ell_\infty$ into their duals, Proc. Amer. Math. Soc., 145 (8) (2017), 3409-3421, doi: 10.1090/proc/13590.
- [**D**] M. Cúth, O. F. K. Kalenda, P. Kaplický: Isometric representation of Lipschitz-free spaces over convex domains in finite-dimensional spaces, Mathematika, 63 (2) (2017), 538–552, doi: 10.1112/S0025579317000031.
- [**E**] M. Cúth, O. F. K. Kalenda, P. Kaplický: Finitely additive measures and complementability of Lipschitz-free spaces, Israel J. Math., 230 (1) (2019), 409–442, doi: 10.1007/s11856-019-1829-y.
- [**F**] L. Candido, M. Cúth, M. Doucha: Isomorphisms between spaces of Lipschitz functions, J. Funct. Anal., 277 (8) (2019), 2697—2727, doi: 10.1016/j.jfa.2019.02.003.
- [**G**] F. Albiac, J. L. Ansorena, M. Cúth, M. Doucha: Lipschitz free $p$-spaces for $0 < p < 1$, accepted in Israel J. Math.
- [**H**] F. Albiac, J. L. Ansorena, M. Cúth, M. Doucha: Embeddability of $\ell_p$ and bases in Lipschitz free $p$-spaces for $0 < p \leq 1$, J. Funct. Anal., in press, doi: 10.1016/j.jfa.2019.108354.

Papers [**A**-**H**] are related to the study of Banach space theoretical properties of "Lipschitz-free Banach spaces". In the first chapter (Introduction) we motivate our research from a personal perspective, we give some basic definitions of the notions we work with and we mention several related areas of mathematics where those or similar classes of spaces are investigated. In the second chapter (Presentation of the included papers) we try to briefly explain the main outcomes of the papers listed above. Reprints of [**A**-**H**] are contained in Appendices.

# 2 Introduction

All of the papers included in this thesis are related to the study of Banach space theoretical properties of “Lipschitz-free Banach spaces”. This class of spaces (or a class of very similar objects) was naturally considered in various areas of research and so we may find different notions describing the same (or very similar) object across more (or sometimes even within one) research fields.[1] In this thesis we use the term “Lipschitz-free” which is due to Godefroy and Kalton [15] and which is frequently used in the community of researchers working in the Banach space theory. Other terms used in Banach space theory for the same object are e.g. “Arens-Eells spaces”, “free Banach spaces” or “transportation cost spaces”.

In this introductory chapter we try to motivate the study of Banach space theoretical properties of Lipschitz-free Banach spaces from the point of view of non-linear geometry of Banach spaces, we give basic definitions of the objects we work with, most importantly we define Lipschitz-free Banach spaces and mention their crucial properties for non-linear geometry of Banach spaces. Finally, we mention several objects with different names (coming often from different areas of mathematics) which are either isometric or very close to Lipschitz-free Banach spaces.

## 2.1 Motivation for non-linear geometry of Banach spaces

Any Banach space $X$ is also a metric space. The principal question of the non-linear geometry of Banach spaces is to find out how much the metric structure of the Banach space $X$ determines its linear structure. One of the first results in the area is the Mazur-Ulam theorem from 1932 [32], by which two Banach spaces are linearly isometric if and only if they are isometric as metric spaces. On the other hand, solving an open problem of Fréchet from 1928, Kadec proved in 1967 [21] that every two separable infinite-dimensional Banach spaces are topologically homeomorphic. Toruńczyk completed Kadec's result in 1981 by proving that every two infinite-dimensional Banach spaces of the same densities are topologically homeomorphic [38].

One of the natural questions which emerged was, what happens if two Banach spaces are Lipschitz-isomorphic (that is, there exists bi-Lipschitz bijection between them). In other words, if $X$ is a Banach space with a property $(P)$ and $Y$ is Banach space which is Lipschitz-isomorphic to $X$, does $Y$ have the property $(P)$ as well? In 1978, Aharoni and Lindenstrauss [1] found a nonseparable Banach space (the space $JL_\infty$) which is not determined by its Lipschitz structure because it is Lipschitz-isomorphic to $c_0 \oplus (JL_\infty/c_0)$ but it is not linearly isomorphic to this space. However, an example of a **separable** Banach space $X$ which is not determined by its Lipschitz structure is not known. This seems to be one of the most important open problems in non-linear geometry of Banach spaces, see e.g. [3, Problem 14.3.1]. It is known that spaces $X = c_0$, $X = \ell_p$ and $X = L_p$ for $1 < p < \infty$ are determined by their Lipschitz structure, but even for classical spaces like $X = \ell_1$, $X = C[0,1]$ or $X = L_1$ it is not known whether there exists a Banach space which is Lipschitz-isomorphic but not linearly isomorphic to $X$, see e.g. the survey [16].

[1] This issue is explained in a greater detail in Section 2.3

In the seminal paper by Godefroy and Kalton [15], the authors considered a construction which assigns to a metric space $M$ a Banach $\mathcal{F}(M)$, called by the authors *Lipschitz-free space over* $M$, in such a way that the linear structure of $\mathcal{F}(M)$ somehow reflects the metric structure of $M$. Using this construction they proved (among others) two very interesting results.

First, they proved that whenever $Q : X \to Y$ is a continuous linear map between separable Banach spaces $X$ and $Y$ (e.g. if $Z \subset X$ is a closed subspace, and $Q : X \to X/Z$ is the quotient map) and $f : Y \to X$ is a $C$-Lipschitz map with $Q \circ f = \mathrm{Id}_Y$ then there is a linear operator $T : Y \to X$ with $Q \circ T = \mathrm{Id}_Y$ and $\|T\| \leq C$. This has two interesting consequences. The first one is that whenever there is an isometric embedding (not necessarily a linear one) of a separable Banach space $X$ into a Banach space $Y$ then $X$ is actually linearly isometric to a subspace of $Y$, see [15, Corollary 3.3]. The second one is that the strategy used by Aharoni and Lindenstrauss in [1] to prove that $JL_\infty$ is not determined by its Lipschitz structure cannot work for separable Banach spaces and therefore, if one wants to construct a separable Banach space not determined by its Lipschitz structure, he/she should come up with a completely new idea (for some details we refer the reader to [3, pages 394-397]).

Second, the authors used Lipschitz-free spaces in order to prove that whenever $X$ and $Y$ are Lipschitz-isomorphic Banach spaces and $X$ has the bounded approximation property (BAP) then $Y$ has BAP as well, see [15, Theorem 5.4].

Soon after the seminal paper by Godefroy and Kalton [15] was published, the study of Lipschitz-free spaces from the point of view of the geometry of Banach spaces became active field of research and many results on their Banach space properties were published. There are still fascinating open problems in the area whose solutions would lead to nice applications, see e.g. [16, Problem 16 and 18]. We should emphasize that even though Lipschitz-free spaces form a class of spaces which is not very difficult to define, the structure of those spaces is still very mysterious, which is witnessed e.g. by the fact that it is not even known whether $\mathcal{F}(\mathbb{R}^2)$ is linearly isomorphic to $\mathcal{F}(\mathbb{R}^3)$. Thus, our understanding of those spaces is far from being satisfactory and there are up to now many papers containing various new structural results. Some of those are mentioned in the next chapter, where we present the content of the papers included in this thesis.

Finally, let us emphasize here that even though we mentioned the paper by Godefroy and Kalton [15] above as the main motivation for the study of Lipschitz free spaces from the point of view of non-linear geometry of Banach spaces, those spaces were essentially known many years before under different names and various interesting results were proved about them. We refer the reader to Section 2.3 where this is explained in a greater detail.

## 2.2 Basic definitions

Let us give some basic definitions and observations which more-or-less follow the approach from Section 1 in [**B**], where we refer the interested reader for the proofs which are considered nowadays as folklore (however they are rather easy). For some basics one may also consult the monograph by Weaver [43].

Let $(M, d, 0)$ be a pointed metric space, that is a metric space with a distinguished "base point" denoted by 0. Consider the space $\mathrm{Lip}_0(M)$ of all real-valued

Lipschitz functions that map $0 \in M$ to $0 \in \mathbb{R}$. It has a vector space structure and the minimal Lipschitz constant of $f \in \mathrm{Lip}_0(M)$ given by

$$\|f\|_{\mathrm{Lip}} := \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x \neq y \in M \right\}$$

gives rise to a norm on $\mathrm{Lip}_0(M)$. Then $\Big( \mathrm{Lip}_0(M), \| \cdot \|_{\mathrm{Lip}} \Big)$ is a Banach space.

For any $x \in M$ we denote by $\delta_x \in \mathrm{Lip}_0(M)^*$ the evaluation functional, that is, $\delta_x(f) = f(x)$ for every $f \in \mathrm{Lip}_0(M)$. Denote by $\mathcal{F}(M)$ the closure of the linear span of $\{\delta_x \colon x \in M\}$ with the dual space norm denoted simply by $\| \cdot \|$. It is easy to see that $\|\delta_x - \delta_y\| = d(x,y)$ for any $x, y \in M$. This means that $M$ can be considered as a metric subspace of $\mathcal{F}(M)$ via the isometric embedding $x \mapsto \delta_x$.

The space $\mathcal{F}(M)$ is in the non-linear geometry of Banach spaces usually called Lipschitz-free space over $M$ and it is uniquely characterized by the following universal property.

**Proposition 1** (Universal property)**.** *Let $(M, d, 0)$ be a pointed metric space. Then $\mathcal{F}(M)$ is the unique (up to isometry) Banach space such that there is an isometry $\delta : M \to \mathcal{F}(M)$ such that*

*(i) $\delta(M \setminus \{0\})$ is linearly independent, and*

*(ii) for every Banach space $Z$ and for every $f \in \mathrm{Lip}_0(M, Z)$ there exists a linear operator $T_f \in \mathcal{L}(\mathcal{F}(M), Z)$ satisfying $T_f \circ \delta = f$ and $\|T_f\| = \|f\|_{\mathrm{Lip}}$.*

Using this universal property of $\mathcal{F}(M)$ for $Z = \mathbb{R}$ it can be rather easily shown that $\mathcal{F}(M)^*$ is linearly isometric to $\mathrm{Lip}_0(M)$. Further, it is easy to observe that it does not matter how the point $0 \in M$ is chosen as the corresponding Lipschitz-free spaces $\mathcal{F}(M)$ are isometric.

Lipschitz free spaces provide a canonical linearization process of Lipschitz maps between metric spaces: if we identify (through the map $\delta$) a metric space $M$ with a subset of $\mathcal{F}(M)$, then any Lipschitz map from a metric space $M$ to a metric space $N$ which maps 0 to 0 extends to a continuous linear map from $\mathcal{F}(M)$ to $\mathcal{F}(N)$. That is, for any $f : M \to N$ with $f(0) = 0$ there exists a linear operator $T_f : \mathcal{F}(M) \to \mathcal{F}(N)$ such that $\|T_f\| = \|f\|_{\mathrm{Lip}}$ and the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle \delta_M}\downarrow & & \downarrow{\scriptstyle \delta_N} \\ \mathcal{F}(M) & \xrightarrow{\quad T_f \quad} & \mathcal{F}(N) \end{array}$$

It is not very difficult to show there are two formulas for the computation of the norm of an element from the canonical countable dense subset. More precisely, given $\mu = \sum_{i=1}^n a_i \delta_{x_i} \in \mathrm{span}\{\delta_x \colon x \in M\} \subset \mathcal{F}(M)$, we have

$$\begin{aligned} \|\mu\|_{\mathcal{F}(M)} &= \sup \left\{ \Big| \sum_{i=1}^n a_i f(x_i) \Big| \colon f \in \mathrm{Lip}_0(M), \|f\|_{\mathrm{Lip}} \leq 1 \right\} \\ &= \inf \left\{ \sum_{j=1}^m |b_j| d(y_j, z_j) \colon \mu = \sum_{j=1}^m b_j (\delta_{y_j} - \delta_{z_j}) \right\}. \end{aligned} \tag{1}$$

Now, one may observe that if $N$ is a subspace of a metric space $M$, then the linear map given by $\mathcal{F}(N) \ni \delta_x \mapsto \delta_x \in \mathcal{F}(M)$, $x \in N$ extends to a linear isometric embedding of $\mathcal{F}(N)$ into $\mathcal{F}(M)$.

Using those observations we can see that the metric structure of $M$ corresponds to the linear structure of $\mathcal{F}(M)$. For example, if $N$ is bi-Lipschitz equivalent (resp. isometric) to a subset of $M$, then $\mathcal{F}(N)$ is linearly isomorphic (resp. linearly isometric) to a subspace of $\mathcal{F}(M)$.

Let us finish this section by an example of an application, which demonstrates the usefulness of Lipschitz-free spaces in the non-linear geometry of Banach spaces. By [15, Theorem 5.3], a Banach space $X$ has the bounded approximation property (BAP) if and only if $\mathcal{F}(X)$ has BAP. A consequence of this result is that BAP is determined by the Lipchitz structure of a Banach space. Indeed, if $X$ has BAP and it is Lipschitz-isomorphic to $Y$, then $\mathcal{F}(X)$ has BAP and it is linearly isomorphic to $\mathcal{F}(Y)$, so $\mathcal{F}(Y)$ has BAP which implies that $Y$ has BAP as well.

## 2.3 Relations to other fields of mathematics

Lipschitz-free spaces are quite natural objects from different perspectives, so it is probably not so big surprise that objects either identical or very close to Lipschitz-free spaces were considered in more areas of mathematics and they were also given different names. Let us describe what is known to us in a greater detail, we refer the interested reader to [41, pages 106-111] and [34, Section 1.6] for some more comments and details concerning historical and terminological remarks.

Let us start with some historical comments concerning the construction provided in Section 2.2. Essentially the same construction has been used already in 1985 by Kadec [22] where the author did not give it a specific name, later in 1986 by Pestov [37] where the author called those spaces *free Banach spaces* and proved those spaces are characterized by the universal property (see Proposition 1) and finally in 2003 by Godefroy and Kalton [15] where the authors called those spaces *Lipschitz-free spaces*.

### 2.3.1 Arens-Eells spaces

Another name under which Lipschitz-free space is well-known among researchers working in the Banach space theory is the *Arens-Eells space*, which is the object considered by Weaver in his monograph [42]. It is easy to see that Lipschitz-free space and Arens-Eells space are naturally isometric as Banach spaces. Let us give some more details.

Let $(M, d)$ be a metric space. In 1956 Arens and Eells [5] considered the vector space $\mathrm{Mol}(M)$ of all the *molecules*, that is, finitely supported functions $m : M \to \mathbb{R}$ satisfying $\sum_{x\in M} m(x) = 0$ endowed with the pseudonorm given by

$$\|m\|_{Æ} := \inf\Big\{\sum_{j=1}^n |a_j| d(x_j, y_j) \colon m = \sum_{j=1}^n a_j(\chi_{x_j} - \chi_{y_j})\Big\}, \quad m \in \mathrm{Mol}(M),$$

where for any $x \in X$, the symbol $\chi_x$ stands for the characteristic function of the singleton set $\{x\}$. They observed that $(\mathrm{Mol}(M), \|\cdot\|_{Æ})$ is a normed linear space and that $M$ isometrically embeds onto a closed subset of $\mathrm{Mol}(M)$.

In his monograph from 1999 Weaver [42] considered the space $\text{Æ}(M)$ which is the completion of $(\text{Mol}(M), \|\cdot\|_{\text{Æ}})$ and he named $\text{Æ}(M)$ the *Arens-Eells space*. From the formula (1) it easily follows that given a base point $0 \in M$, the mapping $\text{Æ}(M) \ni \chi_x - \chi_0 \mapsto \delta_x \in \mathcal{F}(M)$, $x \in M$ extends to the linear isometry between $\text{Æ}(M)$ and $\mathcal{F}(M)$.

Let us note that all the basic properties of Lipschitz-free spaces (equivalently, Arens-Eells spaces) mentioned in Section 2.2 were known to Weaver as well. Detailed proofs are contained in his monograph [42].

There is a natural interpretation of a molecule $m \in \text{Mol}(M)$: $m(x) > 0$ means that $m(x)$ units of a certain product are stored at point $x$; $m(x) < 0$ means that $-m(x)$ units of the same product are needed at $x$. With this in mind, $m \in \text{Mol}(M)$ may be regarded as a transportation problem. If $(a_j)_{j=1}^n$, $(x_j)_{j=1}^n$ and $(y_j)_{j=1}^n$ are such that $m = \sum_{j=1}^n a_j(\chi_{x_j} - \chi_{y_j})$, then we may interpret it as a plan to solve the transportation problem by delivering $a_j$ units from the point $x_j$ to the point $y_j$ for every $j$, the cost of this transportation plan being $\sum_{j=1}^n |a_j| d(x_j, y_j)$. The norm $\|m\|_{\text{Æ}}$ is then interpreted as the minimal possible cost of solving our transportation problem $m$. This is why the space $\text{Æ}(M)$ is also sometimes called the *transportation cost space*, see e.g. [34].

### 2.3.2 Wasserstein distance and Wasserstein spaces

*Wasserstein distance* (or Vasershtein which is the original spelling) is usually though of as a distance between certain probability Borel measures on Polish metric spaces. It is known also as *Kantorovich–Rubinstein distance*, *optimal transportation cost* or *Earth Mover's distance*. It is very questionable what the right notion for this distance should be and who should be attributed for its discovery as it has been discovered and rediscovered by several authors (see [41, pages 106-107] for details). There are also several motivations, one of the most popular ones is to find an optimal solution for various transportation problems, see [40, Introduction].

As mentioned above, the theory is usually developed for Polish metric spaces, see e.g. [40, Section 7.1], but it is possible to generalize the corresponding notions and obtain similar results also for general metric spaces. Let us present the basic notions and results as they are described in [12].

Let $(X, d)$ denote a non-empty metric space, $\mathcal{M}(X)$ be the space of real Radon measures on $X$ and $\mathcal{M}^+(X)$ the set of all the non-negative measures belonging to $\mathcal{M}(X)$. Note that in the literature the authors usually restrict to the situation when $(X, d)$ is separable and complete in which case the class of Radon measures coincides with Borel measures of finite variation, see [7, Theorem 7.1.7]. Denote by $\mathcal{M}_d^+(X)$ the set of all $\mu \in \mathcal{M}^+(X)$ such that for some (and thus for any) $x_0 \in X$ we have

$$\int d(x, x_0)\,\mathrm{d}\mu(x) < +\infty$$

and by $\mathcal{M}_d(X)$ the set of all $\mu \in \mathcal{M}(X)$ such that its total variation $|\mu|$ belongs to $\mathcal{M}_d^+(X)$. Of course, if $d$ is bounded then $\mathcal{M}_d(X) = \mathcal{M}(X)$ and $\mathcal{M}_d^+(X) = \mathcal{M}^+(X)$. Given $\mu, \nu \in \mathcal{M}^+(X)$ with $\mu(X) = \nu(X)$ we denote by

$$\Pi(\mu, \nu) := \{\pi \in \mathcal{M}^+(X \times X) \colon \mu(A) = \pi(A \times X) \text{ and } \nu(A) = \pi(X \times A) \\ \text{for every } A \subset X \text{ Borel}\}$$

the set of all *Radon couplings* of $\mu$ and $\nu$. Given $\mu, \nu \in \mathcal{M}^+(X)$ with $\mu(X) = \nu(X)$ the *Wasserstein distance* between $\mu$ and $\nu$ is defined as

$$W_1(\mu, \nu) := \inf \left\{ \int_{X \times X} d(x, y) \, \mathrm{d}\pi(x, y) \colon \pi \in \Pi(\mu, \nu) \right\}. \tag{2}$$

By the result of Kellerer, see [28, Theorem 1] or a more elementary proof by Edwards [12, Theorem 4.1], the infimum in (2) is attained and we have

$$W_1(\mu, \nu) = \sup \left\{ \int_X f \, \mathrm{d}\mu - \int_X f \, \mathrm{d}\nu \colon f \in \mathrm{Lip}(X), \|f\|_{\mathrm{Lip}} \leq 1 \right\}. \tag{3}$$

It is now rather straightforward to prove that the set of probability measures from $\mathcal{M}_d^+(X)$ endowed with the distance function $W_1$ forms a metric space, see e.g. [12, Theorem 4.4]. This metric space is usually called the *Wasserstein space*, see e.g. [41, Definition 6.4].

The relation with Lipschitz-free spaces is the following. Let us denote by $\mathcal{M}_d^0(X)$ the vector space of all $\mu \in \mathcal{M}_d(X)$ with $\mu(X) = 0$ and put

$$\|\mu\|_W := W_1(\mu^+, \mu^-), \qquad \mu \in \mathcal{M}_d^0(X).$$

Then $(\mathcal{M}_d^0(X), \|\cdot\|_W)$ is a normed linear space, see [12, Theorem 4.5]. By (3), for any base point $0 \in X$ and any $\mu \in \mathcal{M}_d^0(X)$ we have

$$\|\mu\|_W = \sup \left\{ \int_X f \, \mathrm{d}\mu \colon f \in \mathrm{Lip}_0(X), \|f\|_{\mathrm{Lip}} \leq 1 \right\}$$

and so, the mapping $\mathcal{M}_d^0(X) \ni \mu \mapsto T_\mu \in \mathrm{Lip}_0(X)^*$ is isometry where

$$T_\mu(f) = \int_X f \, \mathrm{d}\mu, \quad f \in \mathrm{Lip}_0(X).$$

Moreover, finitely supported measures are dense in $\mathcal{M}_d^0(X)$, see [12, Theorem 6.1], so the completion of $(\mathcal{M}_d^0(X), \|\cdot\|_W)$ is linearly isometric to the Lipschitz-free space $\mathcal{F}(X)$.

Let us conclude this section by mentioning that several authors have published their opinion on the right terminology. The term *Wasserstein distance* and *Wasserstein space* mentioned above was chosen by Villani [41], Vershik [39] provides some reasons for using the notion of *Kantorovich metric*, Weaver in [43, page 125] defends the notion of *Arens-Eells space* and Ostrovskii with Ostrovska in [34, Section 1.6] give reasons to use the term *transportation cost space*.

# 3 Presentation of the included papers

This chapter is devoted to a brief summary of the results included in Appendices, where research articles of the author are presented. We tried to pick the parts which we find the most interesting ones with no attempt to give a complete description of all the papers included. Since some of our ideas from various papers are naturally interconnected, we decided to group our outcomes into three sections according to the topic they cover rather than presenting each one of the papers individually. We hope this will be profitable for the reader as he/she will be able to see more complete picture of what is un/known in the area.

For Banach spaces $X$ and $Y$, we write $X \simeq Y$, $X \hookrightarrow Y$ and $X \overset{c}{\hookrightarrow} Y$ if $X$ is linearly isomorphic to $Y$, $X$ is linearly isomorphic to a subspace of $Y$ and $X$ is linearly isomorphic to a complemented subspace of $Y$, respectively.

## 3.1 Lipschitz free Banach spaces and their $\ell_1$-like behaviour

Soon after the seminal paper by Godefroy and Kalton [15] was published, researchers realized that Lipschitz-free spaces in many respects behave like the space $\ell_1$. As an example we may mention two results by Kalton: $\mathcal{F}(M)$ isomorphically embeds into an infinite $\ell_1$-sum of spaces isomorphic to $\ell_1$ whenever $M$ is a uniformly discrete space (see [24, Proposition 4.3]); $\ell_1(\omega_1) \hookrightarrow \mathcal{F}(M)$ whenever $M$ is a non-separable metric space (see [25, Theorem 2.1]).

One more recent example is the result by Dalet [9] who proved, among other things, that the Lipschitz-free space over a separable proper ultrametric space has the metric approximation property, is isomorphic to $\ell_1$ and is isometric to a dual space. We improved with Doucha [**A**] two of the above-mentioned results.

**Theorem 2** ([**A**, Theorems 1 and 2])**.** *The Lipschitz-free space over a separable infinite ultrametric space is isomorphic to $\ell_1$ and has a monotone Schauder basis.*

Given a separable infinite ultrametric space $M$, we may ask what is the Banach-Mazur distance between $\mathcal{F}(M)$ and $\ell_1$. Dalet, Kaufmann and Procházka [10] proved that $\mathcal{F}(M)$ is not isometric to $\ell_1$. However, in a joint work with Albiac, Ansorena and Doucha [**H**, Proposition 4.5] we proved that the Banach-Mazur distance between $\mathcal{F}(M)$ and $\ell_1$ may be arbitrary close to 1. Since every separable uniformly disconnected metric space is Lipschitz-equivalent to an ultrametric space, we repeated in [**A**] the question of Godefroy whether $\mathcal{F}(K)$ has BAP for every totally disconnected compact metric space $K$. This was answered in negative by Hájek, Lancien and Pernecká [18].

Another class of metric spaces $M$ for which the space $\mathcal{F}(M)$ has quite a lot of common with the space $\ell_1$ are countable compacta. By a result of Dalet [8], $\mathcal{F}(K)$ is a dual space with MAP whenever $K$ is a countable compact metric space. However, with Doucha and Wojtaszczyk [**B**, Theorem 1.2 an Remark 4.5] we proved there exists a sequence $(x_n)$ in $[0,1]^2$ with $x_n \to (0,0)$ such that $\mathcal{F}(\{x_n \colon n \in \mathbb{N}\} \cup \{(0,0)\})$ does not bi-Lipschitz embed into $L_1$.

Another direction of research related to the $\ell_1$-like behaviour of Lipschitz-free spaces was initiated in a joint work with Doucha and Wojtaszczyk [**B**], where we investigated the structure of a general infinite-dimensional Lipschitz-free space. We proved the following.

**Theorem 3** ([**B**, Theorem 1.1])**.** *Let $M$ be an infinite metric space. For the Banach space $X = \mathcal{F}(M)$, we have $\ell_1 \overset{c}{\hookrightarrow} X$. From this we get:*

- *$X^*$ is not separable.*
- *$X$ is not isomorphic to a complemented subspace of a $C(K)$ space.*
- *$X$ is not $\mathcal{L}_\infty$ space.*

This result was noticed by some researchers and cited several times. On the other hand, as we have later found out, the proof is actually quite easy using some classical known facts. Indeed, using [6, Theorem 4] it suffices to show that $\ell_\infty \hookrightarrow X^* = \mathrm{Lip}_0(M)$. Now, picking an infinite disjoint family of non-empty balls in $M$ it is not difficult to construct a sequence of 1-Lipschitz functions (with supports in the balls) equivalent to the $\ell_\infty$-basis. This was pointed out in [19] where the authors generalized our result and proved that $\ell_1(\kappa) \overset{c}{\hookrightarrow} \mathcal{F}(M)$ whenever $\kappa \geq \omega$ is the density of the metric space $M$.

However, our method of the proof of Theorem 3 did not use that much the structure of the dual $X^*$ (see [**B**, Remark 3.3]) and this enabled us to go further. In [**C**] we obtained with Johanis the following improvement.

**Theorem 4** ([**C**, Theorem 5])**.** *Let $M$ be an infinite metric space. Then* $\mathrm{Lip}_0(M)$ *contains a subspace isometric to $\ell_\infty$. If moreover the completion of $M$ has an accumulation point or contains an infinite ultrametric space, then $\mathcal{F}(M)$ contains a* 1*-complemented subspace isometric to $\ell_1$.*

In [**C**] we also formulated the open problem of whether $\ell_1$ embeds isometrically into any infinite-dimensional Lipschitz-free space. This was recently answered in negative by Ostrovska and Ostrovskii [34] and even more recently a characterization of metric spaces $M$ for which $\ell_1$ embeds isometrically into $\mathcal{F}(M)$ was given by the same authors in [35]. Another related recent direction of research is the one from [29] where Khan, Mim and Ostrovskii considered finite-dimensional versions, for example they proved that whenever a metric space $M$ contains $2n$ points then $\mathcal{F}(M)$ contains a 1-complemented subspace isometric to $\ell_1^n$.

With Albiac, Ansorena and Doucha [**H**] we revisited Theorem 3 once more, because we wanted to prove its analogy for "Lipschitz-free $p$-spaces" (see Section 3.3). This resulted in two more generalizations in the setting of classical Lipschitz-free spaces. First, we proved that $\ell_1(\kappa)$ is 3-isomorphic to a 3-complemented subspace of $\mathcal{F}(M)$ whenever $\kappa \geq \omega$ is the density of $M$ (the number 3 is not so important, but the interesting fact is that it does not depend on the metric space $M$, for a more precise and more general statement see Theorem 11 below). Second, we proved the following result.

**Theorem 5** ([**H**, Theorem 3.2])**.** *Let $M$ be an infinite metric space. Then there exists $N \subset M$ such that $\mathcal{F}(N)$ is isomorphic to $\ell_1$ and the canonical copy of $\mathcal{F}(N)$ in $\mathcal{F}(M)$ is complemented in $\mathcal{F}(M)$.*

A natural way of a related possible further research is for example to find out for which nonseparable metric spaces $M$ the space $\ell_1(\mathrm{dens}\, M)$ is isometric to a 1-complemented subspace of $\mathcal{F}(M)$ [**H**, Question 5.2] or to consider the question of whether $\mathcal{F}(K)$ is isomorphic to a dual Banach space whenever $K$ is totally disconnected compact metric space [**A**, Question 1].

Probably the most important question in this area is whether $\mathcal{F}(M)$ has BAP whenever $M$ is a uniformly discrete metric space since both positive and negative answer to this question would have interesting applications, see [16, Problem 18].

## 3.2 Structure of $\mathcal{F}(\mathbb{R}^d)$

Even though it is not difficult to give a definition of a Lipschitz-free space and even though there is an intensive and on-going research concerning their Banach space theoretical properties, we are still far from understanding the structure of those spaces. For example, it is not even known whether $\mathcal{F}(\mathbb{R}^2)$ is isomorphic to $\mathcal{F}(\mathbb{R}^3)$.

It is easy to see that the mapping $\mathrm{Lip}_0(\mathbb{R}) \ni f \mapsto f' \in L_\infty(\mathbb{R})$ is isometry between $\mathrm{Lip}_0(\mathbb{R})$ and $L_\infty(\mathbb{R})$ which is known to be a space with isometrically unique predual. Thus, $\mathcal{F}(\mathbb{R})$ is isometric to $L_1(\mathbb{R})$. On the other hand, using the result of Naor and Schechtman [33], $\mathcal{F}(\mathbb{R}^2)$ does not bi-Lipschitz embed into $L_1(\mathbb{R})$. Thus, at least we know that $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to $\mathcal{F}(\mathbb{R})$.

The study of Banach space theoretical properties of Lipschitz-free spaces over subsets of $\mathbb{R}^d$, $d \in \mathbb{N}$ was given a big attention. Let us summarize some of the most important structural results which were known up to 2015 (more recent contributions are mentioned in the subsequent text).

(i) $L_1 \hookrightarrow \mathcal{F}(M)$ whenever $[0,1]$ bi-Lipschitz embeds into $M$ (easy because $\mathcal{F}([0,1])$ is isometric to $L_1[0,1]$).

(ii) For a metric space $M$, $\mathcal{F}(M)$ isometrically embeds into $L_1$ if and only if $M$ isometrically embeds into an $\mathbb{R}$-tree (see [14]).

(iii) $\mathcal{F}(M)$ has BAP for every $M \subset \mathbb{R}^d$, $d \in \mathbb{N}$. Moreover, if $M$ is compact and convex or $M = \mathbb{R}^d$, then $\mathcal{F}(M)$ has MAP with respect to any norm on $\mathbb{R}^d$ (see [15, 30, 36]).

(iv) $\mathcal{F}(\mathbb{R}^d)$ has a Schauder basis for every $d \in \mathbb{N}$. This basis is even monotone if we endow $\mathbb{R}^d$ with the $\ell_1$-norm (see [20]).

(v) $\mathcal{F}(M)$ is linearly isomorphic to $\mathcal{F}(\mathbb{R}^d)$ which is linearly isomorphic to its $\ell_1$-sum whenever $M \subset \mathbb{R}^d$ has a non-empty interior (see [27]).

In [**D**] with Kalenda and Kaplický we gave an explicit isometric representation of $\mathcal{F}(M)$, where $M$ is a non-empty convex subset in a finite-dimensional normed space.

**Theorem 6** ([**D**, Theorem 1.1]). *Let $E$ be a real normed space of dimension $d \in \mathbb{N}$ and $M \subset E$ be a nonempty convex open subset. Then the Lipschitz-free space $\mathcal{F}(M)$ is canonically isometric to the quotient space*

$$L_1(M,E)/\{\boldsymbol{g} \in L_1(M,E)\colon\ \operatorname{div} \boldsymbol{g} = 0 \text{ in the sense of distributions on } \mathbb{R}^d\}.$$

This result was in part motivated by a result of Lerner who proved that $\mathrm{Lip}_0(\mathbb{R}^d)$ is isometric to the dual of the space described above (his preprint was later included in the paper [17]), but at the time it was not known whether $\mathcal{F}(\mathbb{R}^d)$ is the unique predual of $\mathrm{Lip}_0(\mathbb{R}^d)$ so we had not only to prove that the duals are isometric but we were also led to find out what the mapping $\delta : M \to \mathcal{F}(M)$ is transferred to. Later, Weaver [44] proved that $\mathcal{F}(M)$ is the isometrically unique predual of $\mathrm{Lip}_0(M)$ whenever $M$ is bounded or metrically convex so at least for $M = \mathbb{R}^d$ we can deduce Theorem 6 from the results by Lerner and Weaver mentioned above. A similar characterization of the space $\mathcal{F}(\mathbb{R}^d)$ was also independently given in [13]. As we have learned out later on, similar ideas as we used in our paper [**D**] were used also in [31] in order to characterize Sobolev spaces $W^{-1,1}$ which later motivated us in our work with Albiac, Ansorena and Doucha [**H**] to prove that Sobolev spaces $W^{-1,1}$ are actually isometric to certain Lipschitz-free spaces, see [**H**, Theorem 2.11].

Once we had in hand the isometric characterization of the space $\mathcal{F}(\mathbb{R}^d)$, in a joint work with Kalenda and Kaplický [**E**] we were able to come up with the following result.

**Theorem 7** ([**E**, Theorem 1.1]). *Let $E$ be a normed space of a finite dimension $d \geq 2$. Then there is a linear projection $Q : \mathcal{F}(E)^{**} \to \mathcal{F}(E)$ such that $\|Q\| \leq d_{BM}(E, \ell_d^2)$, where $d_{BM}$ denotes the Banach-Mazur distance.*

One of the motivations for the result above was [16, Problem 16] asking whether $\mathcal{F}(\ell_1)$ is complemented in its bidual. This problem is of particular interest, because a positive answer would solve a famous open problem of whether every Banach space which is Lipschitz-isomorphic to $\ell_1$ is actually linearly isomorphic to $\ell_1$, see [16, comment after Problem 16]. Natural open problems are e.g. whether there exists $C > 0$ such that $\mathcal{F}(E)$ is $C$-complemented in $\mathcal{F}(E)^{**}$ for every finite-dimensional Banach space $E$ or whether $\mathcal{F}(\ell_2)$ is complemented in its bidual, see [**E**, Questions 1.4 and 1.6].

Concerning the problem of whether $\mathcal{F}(\mathbb{R}^2)$ is isomorphic to $\mathcal{F}(\mathbb{R}^3)$ we were thinking about an isomorphisms between the duals and in a joint work with Candido and Doucha [**F**] we proved the following.

**Theorem 8** ([**F**, Theorem 1.16]). $\mathrm{Lip}_0(\mathbb{R}^d) \simeq \mathrm{Lip}_0(\mathbb{Z}^d)$ *for every $d \in \mathbb{N}$.*

On the other hand, $\mathcal{F}(\mathbb{Z}^d)$ is not isomorphic to $\mathcal{F}(\mathbb{R}^d)$ (for example, because $L_1 \hookrightarrow \mathcal{F}(\mathbb{R}^d)$ but $\mathcal{F}(\mathbb{Z}^d)$ is a dual space by the result of Dalet [8]). Thus, one possible way of seeing this result is that it generalizes the known fact that $L_\infty$ is linearly isomorphic to $\ell_\infty$ while $L_1$ is not isomorphic to $\ell_1$, which corresponds to the case of $d = 1$ in Theorem 8.

Moreover, our tools developed in order to prove Theorem 8 had much greater applicability. First, we obtained certain results for "non-commutative finite-dimensional spaces" - in this direction an example of our result is that for quite a general class $\mathcal{C}$ of Carnot groups (for details see [**F**, Theorem 2.2]) whenever $M$ is a member of $\mathcal{C}$ and $N \subset M$ is a net in $M$ (that is, uniformly discrete subset which is $\varepsilon$-dense in $M$ for some $\varepsilon > 0$) then $\mathrm{Lip}_0(M) \simeq \mathrm{Lip}_0(N)$. Next, we considered infinite-dimensional Banach spaces. In this direction we proved e.g. that $\mathrm{Lip}_0(L_p)$ is isomorphic to $\mathrm{Lip}_0(\ell_p)$ or that $\mathrm{Lip}_0(X) \overset{c}{\hookrightarrow} \mathrm{Lip}_0(\mathcal{N}_X)$ whenever $\mathcal{N}_X$ is a net in a Banach space $X$ which is isomorphic to $X \oplus X$ and has a Schauder basis, see [**F**, Section 3] for more details and more results.

Natural open problems in this area of research are for example the following ones. Is $\mathrm{Lip}_0(X)$ isomorphic to $\mathrm{Lip}_0(\mathcal{N}_X)$ whenever $\mathcal{N}_X$ is a net in a separable Banach space $X$? Is it true that $\mathcal{F}(L_p) \simeq \mathcal{F}(\ell_p)$ for every (some) $1 \leq p < \infty$? We refer the reader to [**F**, Section 4] for those and some more natural questions.

## 3.3 Lipschitz-free $p$-spaces

Classes of metric and Banach spaces have natural generalizations.

Fix $0 < p \leq 1$. We say that $(M, d)$ is a *$p$-metric space* if $(M, d^p)$ is a metric space. A *$p$-Banach space* is a vector space $X$ endowed a function $\|\cdot\| : X \to \mathbb{R}$ which satisfies all of the axioms of the norm except that the triangle inequality is replaced by

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p, \quad x, y \in X, \tag{4}$$

such that $(X, \|\cdot\|)$ is complete (that is, when $X$ is endowed with the metric $(x, y) \mapsto \|x - y\|^p$, it is a complete metric space). Of course, the case of $p = 1$ corresponds to the classical notion of a metric space and a Banach space. Let us note that, by the Aoki-Rolewicz theorem, the condition (4) is satisfied if and only if we have $\|x + y\| \leq C(\|x\| + \|y\|)$ for some $0 < C \leq 2^{1/p}$ (see e.g. [26, Theorem 1.3]) which is the reason why $p$-Banach spaces are sometimes given the name of *quasi-Banach spaces*.

It is safe to say that most of the research in functional analysis is done in the framework of Banach spaces (that is, for the case when $p = 1$) and that the study of the more general case of quasi-Banach spaces has lagged far behind despite the fact that the first papers in the subject appeared in the early 1940's, see e.g. [4, 11]. One of the reasons is that for a quasi-Banach space $X$ it might happen that for its dual $X^*$ we have $X^* = \{0\}$ (which is the case e.g. for spaces $L_p[0, 1]$ with $0 < p < 1$) and so working with them requires doing without one of the most powerful tools in Banach spaces: the Hahn-Banach theorem and the duality techniques that rely on it. The reasons which could motivate one to accept the challenge and try to obtain results for the (more demanding) case of a quasi-Banach space are for example the following:

- some people did it and obtained nice results (see e.g. [23]);
- proving new results in $p$-Banach spaces for $0 < p < 1$ often provides an alternative proof even for $p = 1$ and, hopefully, the new techniques developed could give us new observations also for $p = 1$;
- there are many examples of natural quasi-Banach spaces such as the sequence spaces $\ell_p$, the function spaces $L_p$, the Hardy spaces $H_p$, the Lorentz sequence spaces $d(w, p)$, etc.

Similarly as in the setting of metric and Banach spaces, one may consider similar objects to Lipschitz-free Banach spaces in the setting of $p$-metric and $p$-Banach spaces for $0 < p \leq 1$. We have the following analogy to Proposition 1.

**Proposition 9** (Universal property). *Let $0 < p \leq 1$ and $(M, d, 0)$ be a pointed $p$-metric space. Then there exists a unique (up to isometry) $p$-Banach space $\mathcal{F}_p(M)$ such that there is an isometry $\delta : M \to \mathcal{F}_p(M)$ such that* *(i)* $\delta(M \setminus \{0\})$ *is linearly independent, and*

*(ii) for every p-Banach space Z and for every* $f \in \mathrm{Lip}_0(M, Z)$ *there exists a linear operator* $T_f \in \mathcal{L}(\mathcal{F}_p(M), Z)$ *satisfying* $T_f \circ \delta = f$ *and* $\|T_f\| = \|f\|_{\mathrm{Lip}}$.

Lipschitz free $p$-spaces were introduced in [2] with the sole instrumental purpose to build examples for each $0 < p < 1$ of two **separable** $p$-Banach spaces which are Lipschitz-isomorphic but fail to be linearly isomorphic. Whether this is possible or not for $p = 1$ remains as of today the single most important open problem in the theory of non-linear classification of Banach spaces. However, even though Lipschitz free $p$-spaces were proved to be of substantial utility in functional analysis, the structure of those spaces has not been investigated until our joint paper with Albiac, Ansorena and Doucha [**G**] appeared, where we initiated the study of the structure of this new class of $p$-Banach spaces.

In [**G**] we started with some basic observations and proved that all of the basic properties of Lipschitz-free spaces mentioned in Section 2.2 above have their analogies for Lipschitz-free spaces with $0 < p \leq 1$ except for the property that $\mathcal{F}(N)$ is canonically isometric to the subspace of $\mathcal{F}(M)$ whenever $N$ is a subspace of a metric space $M$.

**Theorem 10** ([**G**, Theorem 6.1])**.** *For each* $0 < p < 1$ *and* $p \leq q \leq 1$ *there is a q-metric space* $(M, d)$ *and a subset* $N \subset M$ *such that the inclusion map* $\jmath \colon N \to M$ *induces a non-isometric isomorphic embedding* $L_\jmath : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ *with* $\|L_\jmath^{-1}\| \geq 2^{1/q}$.

It seems to be an important problem whether the canonical mapping $L_\jmath$ from the theorem above must be always an isomorphism, see [**G**, Question 6.2].

On the other hand, not everything is lost. For example, in [**G**] we proved that whenever $M$ is an infinite separable ultrametric space then $\mathcal{F}_p(M)$ is linearly isomorphic to $\ell_p$ for every $0 < p \leq 1$ which generalizes Theorem 2.

In [**H**] we continued in our joint work with Albiac, Ansorena and Doucha. In an analogy to Theorem 7 we proved that, for every $0 < p \leq 1$, $\ell_p$ isomorphically embeds into every infinite-dimensional Lipschitz-free $p$-space (consequently, Lipschitz-free $p$-space is not a $q$-Banach space for $q < p$). Moreover, we proved the following.

**Theorem 11** ([**H**, Theorem 3.1])**.** *Let* $p \in (0, 1]$. *Suppose that* $(M, d)$ *is either*

*(a) a metric space, or*

*(b) a p-metric space containing* $\operatorname{dens} M$*-many isolated points.*

*Then for every* $C > 2^{1/p}$, $\ell_p(\operatorname{dens} M)$ *is C-complemented in* $\mathcal{F}_p(M)$.

Note that in general it is not true that, given $0 < p < 1$, $\ell_p \overset{c}{\hookrightarrow} \mathcal{F}_p(M)$ whenever $M$ is a $p$-metric space as witnessed e.g. by the fact that $\ell_p$ (whose dual is $\ell_\infty$) is not complemented in $L_p[0, 1]$ (which is a Lipschitz-free $p$-space by [**G**, Theorem 4.13]) since $L_p[0, 1]^* = \{0\}$. Let us emphasize that this time our alternative proof for the case of $0 < p < 1$ actually led us to new results even for the classical case of $p = 1$, see e.g. Theorem 5 above.

Moreover, in [**H**] we described the kernel of a projection in a Lipschitz-free $p$-space induced by a Lipschitz retraction. This result seems to be new even for the classical case of $p = 1$.

**Theorem 12** ([**H**, Theorem 2.13]). *Let $(M, d)$ be a pointed $p$-metric space, $0 < p \leq 1$, and $N \subseteq M$ be a Lipschitz retract. Then*

$$\mathcal{F}_p(M) \simeq \mathcal{F}_p(N) \oplus \mathcal{F}_p(M/N),$$

*where $M/N$ denotes the quotient of $M$ by $N$, that is, $M/N = ((M \setminus N) \cup \{0\}, d_{M/N})$ where*

$$d_{M/N}(x, y) := \min\{d(x, y), (d^p(x, N) + d^p(y, N))^{1/p}\}, \quad x, y \in (M \setminus N) \cup \{0\}.$$

Another result from [**H**] which is worth mentioning is that $\mathcal{F}_p([0, 1])$ admits a Schauder basis for every $0 < p \leq 1$. The importance of this result is that it provides up to our knowledge first known examples of $p$-Banach spaces for $p < 1$ with a basis which do not have an unconditional basis and are not a trivial modification/deformation of a Banach space such as $L_1 \oplus \ell_p$, thus reinforcing the theoretical usefulness of Lipschitz free $p$-spaces for $p < 1$.

Let finish this section by mentioning that this work is still in progress and with Albiac, Ansorena and Doucha we have several unpublished results concerning Lipschitz-free $p$-spaces (some of which are new even for the case of $p = 1$) which are very likely to be written carefully down and published later.

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# Appendix A – Lipschitz-free spaces over ultrametric spaces

# Appendix B – Structure of Lipschitz-free spaces

# Appendix C – Isometric embedding of $\ell_1$

# Appendix D – Isometric representation of $\mathcal{F}(\mathbb{R}^d)$

# Appendix E – $\mathcal{F}(\mathbb{R}^d)$ is complemented in its bidual

# Appendix F – Isomorphisms and spaces of Lipschitz functions

# Appendix G – Lipschitz-free $p$-Banach spaces

# Appendix H – Lipschitz-free $p$-Banach spaces II